Non-commutative disintegrations and regular conditional probabilities

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1. Deterministic and nondeterministic processes

2. Stochastic matrices
   - Standard definitions
   - The category of stochastic maps

3. Classical disintegrations
   - Classical disintegrations: intuition
   - Diagrammatic disintegrations
   - Classical disintegrations exist and are unique a.e.

4. Quantum disintegrations
   - Completely positive maps and $\ast$-homomorphisms
   - Non-commutative disintegrations
   - Existence and uniqueness
   - Applications and Examples
Category theory as a theory of processes

Processes can be deterministic or non-deterministic
Deterministic and nondeterministic processes

Category theory as a theory of processes

Processes can be deterministic or non-deterministic

The Kleisli category associated to a monad is one way to distinguish between two such kinds of morphisms.
Goal for non-commutative regular conditional probabilities

Our goal will be to formulate concepts in probability theory categorically. This will enable us to abstract these concepts to contexts beyond their initial domain. We will focus our attention on quantum probability.
Stochastic maps: “if y then x” probabilistic statements

Let $X$ and $Y$ be finite sets. A stochastic map $r : Y \rightsquigarrow X$ assigns a probability measure on $X$ to every point in $Y$. It is a function whose value at a point “spreads out” over the codomain.
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The value $r_y(x)$ of $r_y$ at $x$ is denoted by $r_{xy}$. Since $r_y$ is a probability measure, $r_{xy} \geq 0$ for all $x$ and $y$. Also, $\sum_{x \in X} r_{xy} = 1$ for all $y$. 
Stochastic maps from functions: “if $x$ then $y$” statements

A function $f : X \to Y$ induces a stochastic map $f : X \rightsquigarrow Y$ via

$$f_{yx} := \delta_{yf(x)}$$

where $\delta_{yy'}$ is the Kronecker delta and equals 1 if and only if $y = y'$ and is zero otherwise.
Composing stochastic maps

The composition $\nu \circ \mu : X \rightarrow Z$ of $\mu : X \rightarrow Y$ followed by $\nu : Y \rightarrow Z$ is defined by matrix multiplication

$$(\nu \circ \mu)_{zx} := \sum_{y \in Y} \nu_{zy} \mu_{yx}.$$
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This is completely intuitive! If we start at $x$ and end at $z$, we have the possibility of passing through any intermediate step $y$. These “paths” have associated probabilities, which must be added.
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This is completely intuitive! If we start at $x$ and end at $z$, we have the possibility of passing through any intermediate step $y$. These “paths” have associated probabilities, which must be added.
A probability measure $\mu$ on $X$ can be viewed as a stochastic map $\mu : \{\bullet\} \xrightarrow{\sim} X$ from a single element set.
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Special case: probability measures

- A probability measure $\mu$ on $X$ can be viewed as a stochastic map $\mu : \{\bullet\} \rightarrow X$ from a single element set. Compare this to $\{\bullet\} \rightarrow X$, which picks out a single element of $X$.
- If $f : X \rightarrow Y$ is a function, the composition $f \circ \mu : \{\bullet\} \rightarrow Y$ is the pushforward of $\mu$ along $f$. 
Special case: probability measures

- A probability measure \( \mu \) on \( X \) can be viewed as a stochastic map \( \mu : \{\bullet\} \rightarrow X \) from a single element set. Compare this to \( \{\bullet\} \rightarrow X \), which picks out a single element of \( X \).
- If \( f : X \rightarrow Y \) is a function, the composition \( f \circ \mu : \{\bullet\} \rightarrow Y \) is the pushforward of \( \mu \) along \( f \).
- If \( f : X \rightarrow Y \) is a stochastic map, the composition \( f \circ \mu : \{\bullet\} \rightarrow Y \) is a generalization of the pushforward of a measure. The measure \( f \circ \mu \) on \( Y \) is given by \( (f \circ \mu)(y) = \sum_{x \in X} f_{yx} \mu(x) \) for each \( y \in Y \).
Stochastic maps and their compositions form a category

Composition of stochastic maps is associative and the identity function on any set acts as the identity morphism.
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Composition of stochastic maps is associative and the identity function on any set acts as the identity morphism.
Thus, a commutative diagram of the form

\[
\begin{array}{ccc}
\bullet & \xrightarrow{\mu} & X \\
\downarrow\mu & & \downarrow f \\
\downarrow\nu & & \downarrow Y \\
\end{array}
\]

says that $\mu$ is a probability measure on $X$ and its pushforward to $Y$ along $f$ is the probability measure $\nu$. 
A disintegration is a stochastic section

Let $X$ and $Y$ be finite sets equipped with probability measures. Gromov pictures a measure-preserving function $f : X \to Y$ in terms of water droplets. $f$ combines the water droplets and their volume (probabilities) add when they combine under $f$. 
A disintegration is a stochastic section

Let $X$ and $Y$ be finite sets equipped with probability measures. Gromov pictures a measure-preserving function $f : X \rightarrow Y$ in terms of water droplets. $f$ combines the water droplets and their volume (probabilities) add when they combine under $f$. **A disintegration** $r : Y \rightsquigarrow X$ is a measure-preserving stochastic section of $f$. 
Disintegrations: diagrammatic definition

Definition

Let $(X, \mu)$ and $(Y, \nu)$ be probability spaces and let $f : X \rightarrow Y$ be a function such that the diagram on the right commutes.

A disintegration of $(f, \mu, \nu)$ is a stochastic map $Y \rightarrow X$ such that

\[ \begin{array}{c}
X \\
\downarrow \mu \\
\downarrow f \\
\downarrow \nu \\
Y \\
\end{array} \]

the latter diagram signifying commutativity $\nu$-a.e.

A disintegration is also called a regular conditional probability and an optimal hypothesis.
**Disintegrations: diagrammatic definition**

**Definition**

Let \((X, \mu)\) and \((Y, \nu)\) be probability spaces and let \(f : X \rightarrow Y\) be a function such that the diagram on the right commutes.

![Diagram of commutative diagram]

A **disintegration** of \((f, \mu, \nu)\) is a stochastic map \(Y \xrightarrow{r} X\) such that the latter diagram signifying commutativity \(\nu\text{-a.e.}

A disintegration is also called a **regular conditional probability** and an **optimal hypothesis**.
Classical disintegrations exist and are unique a.e. (almost everywhere).
Classical disintegrations exist and are unique a.e. (almost everywhere). That’s really all you need to know!
Application: Bayes’ theorem

Question: Where do disintegrations show up?
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Answer: statistical inference!
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Corollary (Bayes’ theorem)

Given \( \{ \bullet \} \xrightarrow{p} X \xrightarrow{f} Y \), there exists a \( Y \xrightarrow{g} X \) such that

\[
\begin{align*}
\text{Y} & \xleftarrow{f \circ p} \{ \bullet \} \xrightarrow{p} \text{X} \\
\text{Y} \times \text{Y} & \xleftarrow{g \times \text{id}_Y} \text{X} \times \text{Y} \xleftarrow{\text{id}_X \times f} \text{X} \times \text{X}
\end{align*}
\]

and

\[
\begin{align*}
\text{Y} & \xleftarrow{\Delta_Y} \{ \bullet \} \xrightarrow{p} \text{X} \\
\text{Y} \times \text{Y} & \xleftarrow{g \times \text{id}_Y} \text{X} \times \text{Y} \xleftarrow{\text{id}_X \times f} \text{X} \times \text{X}
\end{align*}
\]

Furthermore, for any other \( g' \) satisfying these two conditions, \( g \xrightarrow{f \circ p} g' \).
Application: Bayes’ theorem

Proof.

Take $g$ to be the composition $Y \xrightarrow{h} X \times Y \xrightarrow{\pi_X} X$, 
Application: Bayes’ theorem

Proof.

Take \( g \) to be the composition \( Y \xrightarrow{h} X \times Y \xrightarrow{\pi_X} X \), where \( h \) is a disintegration of

\[
\begin{array}{c}
\{ \bullet \} \\
p \\
\Delta_X \\
\end{array}
\xrightarrow{p} 
\begin{array}{c}
X \\
\Delta_X \\
\end{array}
\xrightarrow{f \circ p} 
\begin{array}{c}
X \times X \\
id_X \times f \\
\end{array}
\xrightarrow{\pi_Y} 
\begin{array}{c}
X \times Y \\
Y \\
\end{array}
\]
Objects: Finite-dimensional $C^*$-algebras

Let $\mathcal{M}_n(\mathbb{C})$ denote the set of complex $n \times n$ matrices. It is an example of a $C^*$-algebra: we can add and multiply $n \times n$ matrices, the operator norm gives a norm, and $A^*$ is the conjugate transpose of $A$. 
Objects: Finite-dimensional $C^*$-algebras

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- Every finite-dimensional $C^*$-algebra is ($C^*$-algebraically isomorphic to) a direct sum of matrix algebras.
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- Every finite-dimensional $C^*$-algebra is ($C^*$-algebraically isomorphic to) a direct sum of matrix algebras.

- In particular, $\mathbb{C}^X$, functions from a finite set $X$ to $\mathbb{C}$, is a commutative $C^*$-algebra (it is isomorphic to $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$). A basis for this algebra as a vector space is $\{e_x\}_{x \in X}$ defined by $e_x(x') := \delta_{xx'}$. 

Morphisms: $\ast$-homomorphisms and CPU maps

- Every *completely positive unital (CPU) map* $\varphi: \mathcal{M}_m(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C})$ preserves positivity of matrices and their tensor products with finite-dimensional identities.
Morphisms: $*$-homomorphisms and CPU maps

- Every completely positive unital (CPU) map $\varphi : \mathcal{M}_m(\mathbb{C}) \xrightarrow{\sim} \mathcal{M}_n(\mathbb{C})$ preserves positivity of matrices and their tensor products with finite-dimensional identities.
- Every (unital) $*$-homomorphism $F : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ is of the form

  $$F(A) = U \begin{bmatrix} A & 0 \\ \cdot & \cdot \\ 0 & A \end{bmatrix} U^*,$$

  where $U$ is unitary. In particular $m = np$ for some $p \in \mathbb{N}$. 
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  \]
  where $U$ is unitary. In particular $m = np$ for some $p \in \mathbb{N}$.
- For every CPU map $\omega : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathbb{C}$ (called a state), there exists a unique $n \times n$ positive matrix $\rho$ such that $\text{tr}(\rho) = 1$ and $\text{tr}(\rho A) = \omega(A)$ for all $A \in \mathcal{M}_n(\mathbb{C})$. $\rho$ is called a density matrix.
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- Every **completely positive unital (CPU) map** $\varphi : \mathcal{M}_m(\mathbb{C}) \rightarrow \mathcal{M}_n(\mathbb{C})$ preserves positivity of matrices and their tensor products with finite-dimensional identities.

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- For every CPU map $\omega : \mathbb{C}^X \rightarrow \mathbb{C}$ (also called a **state**), there exists a unique probability measure $p : \{\bullet\} \rightarrow X$ such that $\omega(\varphi) = \sum_{x \in X} p_x \varphi(x)$ for all $\varphi \in \mathbb{C}^X$. We write this state as $\langle p, \cdot \rangle$. 
From finite sets to finite-dimensional $C^*$-algebras

There is a (contravariant) fully faithful functor from finite sets and stochastic maps to finite-dimensional $C^*$-algebras and CPU maps.

<table>
<thead>
<tr>
<th>category theory</th>
<th>classical/commutative</th>
<th>quantum/noncommutative</th>
<th>physics/interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>object</td>
<td>set</td>
<td>$C^*$-algebra</td>
<td>phase space observables</td>
</tr>
<tr>
<td>$\rightarrow$ morphism</td>
<td>function</td>
<td>$*$-homomorphism</td>
<td>deterministic process</td>
</tr>
<tr>
<td>$\leadsto$ morphism</td>
<td>stochastic map</td>
<td>CPU map</td>
<td>non-deterministic process</td>
</tr>
<tr>
<td>monoidal product</td>
<td>cartesian product $\times$</td>
<td>tensor product $\otimes$</td>
<td>combining systems</td>
</tr>
<tr>
<td>$\leadsto$ to/from monoidal unit</td>
<td>probability measure</td>
<td>$C^*$-algebra state/density matrix</td>
<td>physical state</td>
</tr>
</tbody>
</table>
Definition (P–Russo)

Let $(\mathcal{A}, \omega)$ and $(\mathcal{B}, \xi)$ be $C^*$-algebras equipped with states. Let $F : \mathcal{B} \to \mathcal{A}$ be a $*$-homomorphism such that the diagram on the right commutes.
Definition (P–Russo)

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A \textit{disintegration} of \(\omega\) over \(\xi\) consistent with \(F\) is a CPU map \(R : \mathcal{A} \frown \mathcal{B}\) such that

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{R} & \mathcal{B} \\
\xleftarrow{\omega} & & \xleftarrow{\xi} \\
\xrightarrow{\mathcal{C}} & & \xrightarrow{\mathcal{C}}
\end{array}
\]

and

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{id_B} & \mathcal{B} \\
\xleftarrow{F} & & \xleftarrow{R} \\
\xrightarrow{\xi} & & \xrightarrow{\mathcal{C}}
\end{array}
\]

the latter diagram signifying commutativity \(\xi\)-a.e.
Existence and uniqueness of disintegrations

Surprising: existence is not guaranteed in the non-commutative setting!
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Surprising: existence is **not guaranteed** in the non-commutative setting!

**Theorem (P–Ruso)**

Fix \( n, p \in \mathbb{N} \). Let

\[
\begin{align*}
\mathcal{M}_{np}(\mathbb{C}) & \xleftarrow{F} \mathcal{M}_n(\mathbb{C}) \\
\text{tr}(\rho \cdot ) & \equiv \omega \\
\xi & \equiv \text{tr}(\sigma \cdot )
\end{align*}
\]

be a commutative diagram with \( F \) the \(*\)-homomorphism given by the block diagonal inclusion \( F(A) = \text{diag}(A, \ldots, A) \).
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\]

be a commutative diagram with $F$ the $\ast$-homomorphism given by the block diagonal inclusion $F(A) = \text{diag}(A, \ldots, A)$. A disintegration of $\omega$ over $\xi$ consistent with $F$ exists if and only if there exists a density matrix $\tau \in \mathcal{M}_p(\mathbb{C})$ such that $\rho = \tau \otimes \sigma$. 
Example 1: Einstein–Podolsky–Rosen

**Theorem (P–Russo)**

Let

\[
\rho := \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\quad \& \quad
\sigma := \frac{1}{2} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}.
\]

and let \( F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C}) \) be the diagonal map.
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\]

and let \( F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C}) \) be the diagonal map. Then

\[
\text{tr}(\sigma A) = \text{tr}(\rho F(A))
\]

for all \( A \) but there does not exist a disintegration of \( \rho \) over \( \sigma \) consistent with \( F \).
Example 1: Einstein–Podolsky–Rosen

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Let

$$\rho := \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \& \quad \sigma := \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. $$

and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ be the diagonal map. Then $\text{tr}(\sigma A) = \text{tr}(\rho F(A))$ for all $A$ but there does not exist a disintegration of $\rho$ over $\sigma$ consistent with $F$.

Proof.

$\rho$ is entangled (not separable) and therefore cannot be expressed as the tensor product of any two $2 \times 2$ density matrices.
Example 2: Diagonal density matrices

**Theorem (P–Russo)**

Fix \( p_1, p_2, p_3, p_4 \geq 0 \) with \( p_1 + p_2 + p_3 + p_4 = 1 \), \( p_1 + p_3 > 0 \), and \( p_2 + p_4 > 0 \). Let

\[
\rho = \begin{bmatrix}
 p_1 & 0 & 0 & 0 \\
 0 & p_2 & 0 & 0 \\
 0 & 0 & p_3 & 0 \\
 0 & 0 & 0 & p_4 \\
\end{bmatrix}
\]

\&

\[
\sigma = \begin{bmatrix}
 p_1 + p_3 & 0 \\
 0 & p_2 + p_4 \\
\end{bmatrix}
\]

be density matrices and let \( F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C}) \) be the block diagonal inclusion.

Furthermore, there exists a disintegration of \( \rho \) over \( \sigma \) consistent with \( F \) if and only if \( p_1 p_4 = p_2 p_3 \).
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$$

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$$
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$$
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    0 & p_2 & 0 & 0 \\
    0 & 0 & p_3 & 0 \\
    0 & 0 & 0 & p_4
\end{bmatrix} \quad \& \quad \sigma = \begin{bmatrix}
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$$

be density matrices and let $F : \mathcal{M}_2(\mathbb{C}) \rightarrow \mathcal{M}_4(\mathbb{C})$ be the block diagonal inclusion. Then $\text{tr}(\sigma A) = \text{tr}(\rho F(A))$ for all $A$. Furthermore, there exists a disintegration of $\rho$ over $\sigma$ consistent with $F$ if and only if

$$
p_1p_4 = p_2p_3.
$$
Example 3: Measurement in quantum mechanics

Theorem (P–Russo)

Let $A \in \mathcal{M}_m(\mathbb{C})$ be a self-adjoint matrix with spectrum $\sigma(A)$,
Example 3: Measurement in quantum mechanics

Theorem (P–Russo)

Let \( A \in \mathcal{M}_m(\mathbb{C}) \) be a self-adjoint matrix with spectrum \( \sigma(A) \), let \( F : \mathbb{C}^{\sigma(A)} \to \mathcal{M}_m(\mathbb{C}) \) be the unique \(*\)-homomorphism determined by

\[
\mathbb{C}^{\sigma(A)} \xrightarrow{F} \mathcal{M}_m(\mathbb{C})
\]

\[
e_\lambda \mapsto P_\lambda,
\]

where the right-hand-side is called the Lüders projection of \( \rho \) with respect to the measurement of \( A \).
Example 3: Measurement in quantum mechanics

Theorem (P–Russo)

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\[
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\[
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\]

and let \( \omega = \text{tr}(\rho \cdot) : \mathcal{M}_m(\mathbb{C}) \twoheadrightarrow \mathbb{C} \) be a state with \( \langle q, \cdot \rangle := \omega \circ F \) the induced state on \( \mathbb{C}^{\sigma(A)} \).
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Let \( A \in \mathcal{M}_m(\mathbb{C}) \) be a self-adjoint matrix with spectrum \( \sigma(A) \), let \( F : \mathbb{C}^{\sigma(A)} \to \mathcal{M}_m(\mathbb{C}) \) be the unique \(*\)-homomorphism determined by

\[
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e_\lambda \mapsto P_\lambda,
\]

and let \( \omega = \text{tr}(\rho \cdot) : \mathcal{M}_m(\mathbb{C}) \xrightarrow{\sim} \mathbb{C} \) be a state with \( \langle q, \cdot \rangle := \omega \circ F \) the induced state on \( \mathbb{C}^{\sigma(A)} \). Then \( F \) has a disintegration if and only if

\[
\rho = \sum_{\lambda \in \sigma(A)} P_\lambda \rho P_\lambda,
\]

where the right-hand-side is called the \textbf{Lüders projection} of \( \rho \) with respect to the measurement of \( A \).
Example 4: A “no-go” theorem for pure to mixed states

There are no disintegrations for evolving pure states to mixed states (a state is \textit{pure} iff it is an extreme point of the convex set of states).
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**Theorem (P–Russo)**

*Given a commutative diagram* 

\[ \mathcal{M}_{np}(\mathbb{C}) \xleftarrow{F} \mathcal{M}_n(\mathbb{C}) \]

\[ \text{tr}(\rho \cdot ) \quad \text{tr}(\sigma \cdot ) \]

\[ \mathbb{C} \]

*of CPU maps with \( \rho \) pure, if a disintegration exists, then \( \sigma \) must necessarily be pure as well.*
Thank you!

Thank you for your attention!