Noncommutative disintegration

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1 Deterministic and nondeterministic processes

2 Stochastic matrices
   - Standard definitions
   - The category of stochastic maps

3 Classical disintegrations
   - Classical disintegrations: intuition
   - Diagrammatic disintegrations
   - Classical disintegrations exist and are unique a.e.

4 Quantum disintegrations
   - Completely positive maps and $*$-homomorphisms
   - Non-commutative disintegrations
   - Existence and uniqueness
   - Examples
Category theory as a theory of processes

Processes can be deterministic or non-deterministic
Stochastic maps

Let $X$ and $Y$ be finite sets. A stochastic map $r : Y \rightsquigarrow X$ assigns a probability measure on $X$ to every point in $Y$. It is a function whose value at a point “spreads out” over the codomain.
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Let $X$ and $Y$ be finite sets. A **stochastic map** $r : Y \rightsquigarrow X$ assigns a probability measure on $X$ to every point in $Y$. It is a function whose value at a point “spreads out” over the codomain.

The value $r_y(x)$ of $r_y$ at $x$ is denoted by $r_{xy}$. Since $r_y$ is a probability measure, $r_{xy} \geq 0$ for all $x$ and $y$. Also, $\sum_{x \in X} r_{xy} = 1$ for all $y$. 
A function \( f : X \to Y \) induces a stochastic map \( \tilde{f} : X \sim \to Y \) via

\[
f_{yx} := \delta_y f(x)
\]

where \( \delta_{yy'} \) is the Kronecker delta and equals 1 if and only if \( y = y' \) and is zero otherwise.
Composing stochastic maps

The **composition** $\nu \circ \mu : X \rightsquigarrow Z$ of $\mu : X \rightsquigarrow Y$ followed by $\nu : Y \rightsquigarrow Z$ is defined by matrix multiplication

$$(\nu \circ \mu)_{zx} := \sum_{y \in Y} \nu_{zy} \mu_{yx}.$$
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This is completely intuitive! If we start at $x$ and end at $z$, we have the possibility of passing through any intermediate step $y$. These “paths” have associated probabilities, which must be added.
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Special case: probability measures

- A probability measure $\mu$ on $X$ can be viewed as a stochastic map $\mu : \{\bullet\} \rightarrowto X$ from a single element set.
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- A probability measure $\mu$ on $X$ can be viewed as a stochastic map $\mu : \{\bullet\} \rightsquigarrow X$ from a single element set.
- If $f : X \to Y$ is a function, the composition $f \circ \mu : \{\bullet\} \rightsquigarrow Y$ is the pushforward of $\mu$ along $f$. 
Special case: probability measures

- A probability measure $\mu$ on $X$ can be viewed as a stochastic map $\mu : \{\bullet\} \rightarrow X$ from a single element set.
- If $f : X \rightarrow Y$ is a function, the composition $f \circ \mu : \{\bullet\} \rightarrow Y$ is the pushforward of $\mu$ along $f$.
- If $f : X \rightarrow Y$ is a stochastic map, the composition $f \circ \mu : \{\bullet\} \rightarrow Y$ is a generalization of the pushforward of a measure. The measure $f \circ \mu$ on $Y$ is given by $(f \circ \mu)(y) = \sum_{x \in X} f_{yx} \mu(x)$ for each $y \in Y$. 
Stochastic matrices and their compositions form a category

Composition of stochastic maps is associative and the identity function on any set acts as the identity morphism.
Disintegrations as a section

Gromov pictures a measure-preserving function $f : X \to Y$ in terms of water droplets. $f$ combines the water droplets and their volume (probabilities) add when they combine under $f$. 
Disintegrations as a section

Gromov pictures a measure-preserving function $f : X \to Y$ in terms of water droplets. $f$ combines the water droplets and their volume (probabilities) add when they combine under $f$. A disintegration $r : Y \rightsquigarrow X$ is a measure-preserving stochastic section of $f$. 
Definition

Let \((X, \mu)\) and \((Y, \nu)\) be probability spaces and let \(f : X \to Y\) be a function such that the diagram on the right commutes.
Disintegrations: diagrammatic definition

**Definition**

Let \((X, \mu)\) and \((Y, \nu)\) be probability spaces and let \(f : X \to Y\) be a function such that the diagram on the right commutes.

A **disintegration** of \(\mu\) over \(\nu\) consistent with \(f\) is a stochastic map \(r : Y \leadsto X\) such that

\[
\begin{align*}
X & \xrightarrow{\mu} Y \\
Y & \xleftarrow{\nu} X \\
Y & \xleftarrow{id_Y}
\end{align*}
\]

and

the latter diagram signifying commutativity \(\nu\text{-a.e.}\).
Theorem

Let \((X, \mu)\) and \((Y, \nu)\) be finite sets equipped with probability measures \(\mu\) and \(\nu\). Let \(f : X \to Y\) be a measure-preserving function. Then there exists a unique (\(\nu\)-a.e.) disintegration \(r : Y \rightharpoonup X\) of \(\mu\) over \(\nu\) consistent with \(f\).
Classical disintegrations exist and are unique a.e.

**Theorem**

Let $(X, \mu)$ and $(Y, \nu)$ be finite sets equipped with probability measures $\mu$ and $\nu$. Let $f : X \to Y$ be a measure-preserving function. Then there exists a unique ($\nu$-a.e.) disintegration $r : Y \rightrightarrows X$ of $\mu$ over $\nu$ consistent with $f$.

In fact, a formula for the disintegration is

$$r_{xy} := \begin{cases} \frac{\mu_x \delta_y f(x)}{\nu_y} & \text{if } \nu_y > 0 \\ \frac{1}{|X|} & \text{otherwise} \end{cases}$$
Matrix algebras

- Let $\mathcal{M}_n(\mathbb{C})$ denote the set of complex $n \times n$ matrices. It is an example of a $C^*$-algebra: we can add and multiply $n \times n$ matrices, the operator norm gives a norm, and $A^*$ is the conjugate transpose of $A$. 
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- Every finite-dimensional $C^*$-algebra is ($C^*$-algebraically isomorphic to) a direct sum of matrix algebras.
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In particular, $\mathbb{C}^X$, functions from a finite set $X$ to $\mathbb{C}$, is a commutative $C^*$-algebra. A basis for this algebra as a vector space is $\{e_x\}_{x \in X}$ defined by $e_x(x') := \delta_{xx'}$. 
Matrix algebras

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- Every finite-dimensional $C^*$-algebra is ($C^*$-algebraically isomorphic to) a direct sum of matrix algebras.
- In particular, $\mathbb{C}^X$, functions from a finite set $X$ to $\mathbb{C}$, is a commutative $C^*$-algebra. A basis for this algebra as a vector space is $\{e_x\}_{x \in X}$ defined by $e_x(x') := \delta_{xx'}$.
- If $\mathcal{A}$ is a $C^*$-algebra, then $\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{A}$ can be viewed as $n \times n$ matrices with entries in $\mathcal{A}$. It has a natural $C^*$-algebra structure.
Completely positive maps and $\ast$-homomorphisms

**Definition**

Let $A$ and $B$ be finite-dimensional $C^*$-algebras with units $1_A$ and $1_B$ (think direct sums of matrix algebras).
Completely positive maps and $*$-homomorphisms

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Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional $C^*$-algebras with units $1_\mathcal{A}$ and $1_\mathcal{B}$ (think direct sums of matrix algebras). An element of a $C^*$-algebra $\mathcal{A}$ is \textit{positive} iff it equals $a^*a$ for some $a \in \mathcal{A}$. 
Completely positive maps and \(*\)-homomorphisms

**Definition**

Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional $C^*$-algebras with units $1_\mathcal{A}$ and $1_\mathcal{B}$ (think direct sums of matrix algebras). An element of a $C^*$-algebra $\mathcal{A}$ is **positive** iff it equals $a^*a$ for some $a \in \mathcal{A}$. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is **positive** iff it sends positive elements to positive elements.
Completely positive maps and \( \ast \)-homomorphisms

**Definition**

Let \( A \) and \( B \) be finite-dimensional \( C^\ast \)-algebras with units \( 1_A \) and \( 1_B \) (think direct sums of matrix algebras). An element of a \( C^\ast \)-algebra \( A \) is **positive** iff it equals \( a^*a \) for some \( a \in A \). A linear map \( \varphi : A \rightarrow B \) is **positive** iff it sends positive elements to positive elements. A linear map \( \varphi : A \rightarrow B \) is **\( n \)-positive** iff \( \text{id}_{M_n(\mathbb{C})} \otimes \varphi : M_n(\mathbb{C}) \otimes A \rightarrow M_n(\mathbb{C}) \otimes B \) is positive.
Completely positive maps and ∗-homomorphisms

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional $C^*$-algebras with units $1_\mathcal{A}$ and $1_\mathcal{B}$ (think direct sums of matrix algebras). An element of a $C^*$-algebra $\mathcal{A}$ is positive iff it equals $a^*a$ for some $a \in \mathcal{A}$. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is positive iff it sends positive elements to positive elements. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is $n$-positive iff $\text{id}_{M_n(\mathbb{C})} \otimes \varphi : M_n(\mathbb{C}) \otimes A \to M_n(\mathbb{C}) \otimes B$ is positive. $\varphi$ is completely positive iff $\varphi$ is $n$-positive for all $n \in \mathbb{N}$. 
Completely positive maps and $\ast$-homomorphisms

**Definition**

Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional $C^*$-algebras with units $1_\mathcal{A}$ and $1_\mathcal{B}$ (think direct sums of matrix algebras). An element of a $C^*$-algebra $\mathcal{A}$ is *positive* iff it equals $a^*a$ for some $a \in \mathcal{A}$. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is *positive* iff it sends positive elements to positive elements. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is *n-positive* iff $\text{id}_{M_n(\mathbb{C})} \otimes \varphi : M_n(\mathbb{C}) \otimes \mathcal{A} \to M_n(\mathbb{C}) \otimes \mathcal{B}$ is positive. $\varphi$ is *completely positive* iff $\varphi$ is $n$-positive for all $n \in \mathbb{N}$. A $\ast$-homomorphism $\mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to $\mathcal{B}$ is a function preserving the $C^*$-algebra structure: $f$ is linear, $f(aa') = f(a)f(a')$, $f(1_\mathcal{A}) = 1_\mathcal{B}$, and $f(a^*) = f(a)^*$. 
Completely positive maps and $*$-homomorphisms

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Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional $C^*$-algebras with units $1_\mathcal{A}$ and $1_\mathcal{B}$ (think direct sums of matrix algebras). An element of a $C^*$-algebra $\mathcal{A}$ is **positive** iff it equals $a^*a$ for some $a \in \mathcal{A}$. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is **positive** iff it sends positive elements to positive elements. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is **$n$-positive** iff $\text{id}_{M_n(\mathbb{C})} \otimes \varphi : M_n(\mathbb{C}) \otimes \mathcal{A} \to M_n(\mathbb{C}) \otimes \mathcal{B}$ is positive. $\varphi$ is **completely positive** iff $\varphi$ is $n$-positive for all $n \in \mathbb{N}$. A **$*$-homomorphism** $\mathcal{A} \to \mathcal{B}$ from $\mathcal{A}$ to $\mathcal{B}$ is a function preserving the $C^*$-algebra structure: $f$ is linear, $f(aa') = f(a)f(a')$, $f(1_\mathcal{A}) = 1_\mathcal{B}$, and $f(a^*) = f(a)^*$. A positive unital map $\mathcal{A} \to \mathbb{C}$ is called a **state**.
Examples

- An $n \times n$ matrix is positive if and only if it is self-adjoint and its eigenvalues are non-negative.
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- An $n \times n$ matrix is positive if and only if it is self-adjoint and its eigenvalues are non-negative.

- A $\ast$-homomorphism $F : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_m(\mathbb{C})$ exists if and only if $m = np$ for some $p \in \mathbb{N}$. When this happens, there exists a unitary $m \times m$ matrix $U$ (unitary means $UU^* = 1_m$) such that

$$F(A) = U \begin{bmatrix} A & 0 \\ \vdots & \ddots \\ 0 & A \end{bmatrix} U^*$$

for all $A \in \mathcal{M}_n(\mathbb{C})$. 
Examples

- An $n \times n$ matrix is positive if and only if it is self-adjoint and its eigenvalues are non-negative.
- A $*$-homomorphism $F : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathcal{M}_m(\mathbb{C})$ exists if and only if $m = np$ for some $p \in \mathbb{N}$. When this happens, there exists a unitary $m \times m$ matrix $U$ (unitary means $UU^* = 1_m$) such that
  \[ F(A) = U \begin{bmatrix} A & 0 \\ \vdots & \ddots \\ 0 & A \end{bmatrix} U^* \quad \text{for all } A \in \mathcal{M}_n(\mathbb{C}). \]
- If $\omega : \mathcal{M}_n(\mathbb{C}) \rightarrow \mathbb{C}$ is a state, there exists a unique $n \times n$ positive matrix $\rho$ such that $\text{tr}(\rho) = 1$ and $\text{tr}(\rho A) = \omega(A)$ for all $A \in \mathcal{M}_n(\mathbb{C})$. 
There is a (contravariant) functor from finite sets and stochastic maps to finite-dimensional $C^*$-algebras and completely positive maps.

<table>
<thead>
<tr>
<th>category theory</th>
<th>classical/commutative</th>
<th>quantum/noncommutative</th>
<th>physics/interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>object</td>
<td>set</td>
<td>$C^*$-algebra</td>
<td>phase space observables</td>
</tr>
<tr>
<td>$\rightarrow$ morphism</td>
<td>function</td>
<td>$\ast$-homomorphism</td>
<td>deterministic process</td>
</tr>
<tr>
<td>$\sim\rightarrow$ morphism</td>
<td>stochastic map</td>
<td>completely positive map</td>
<td>non-deterministic process</td>
</tr>
<tr>
<td>monoidal product</td>
<td>cartesian product $\times$</td>
<td>tensor product $\otimes$</td>
<td>combining systems</td>
</tr>
<tr>
<td>$\sim\rightarrow$ to/from monoidal unit</td>
<td>probability measure</td>
<td>$C^*$-algebra state/density matrix</td>
<td>physical state</td>
</tr>
</tbody>
</table>
From finite sets to finite-dimensional $C^*$-algebras II

Briefly, this functor is given by

$$X \mapsto \mathbb{C}^X$$

$$(f : X \rightarrow Y) \mapsto \left( \mathbb{C}^Y \ni e_y \mapsto \sum_{x \in X} f_{yx} e_x \in \mathbb{C}^X \right)$$

Therefore, an arbitrary function $\phi = \sum_{y \in Y} \phi(y) e_y \in \mathbb{C}^Y$ gets sent to

$$\sum_{y \in Y} \phi(y) \sum_{x \in X} f_{yx} e_x = \sum_{x \in X} \phi(f(x)) e_x = \phi \circ f$$

the pullback of $\phi$ along $f$. 
From finite sets to finite-dimensional $C^*$-algebras II

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$$(f : X \rightarrow Y) \mapsto \left( \mathbb{C}^Y \ni e_y \mapsto \sum_{x \in X} f_{yx} e_x \in \mathbb{C}^X \right)$$

In the special case where $f$ is a $*$-homomorphism, $f_{yx} = \delta_{yf(x)}$, the sum reduces to

$$\sum_{x \in X} f_{yx} e_x = \sum_{x \in X} \delta_{yf(x)} e_x = \sum_{x \in f^{-1}(y)} e_x$$
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In the special case where \( f \) is a *-homomorphism, \( f_{yx} = \delta_{yf(x)} \), the sum reduces to

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Therefore, an arbitrary function \( \varphi = \sum_{y \in Y} \varphi(y) e_y \in \mathbb{C}^Y \) gets sent to

\[ \sum_{y \in Y} \sum_{x \in X} f_{yx} e_x = \sum_{y \in Y} \sum_{x \in f^{-1}(y)} \varphi(y) e_x = \sum_{x \in X} \varphi(f(x)) e_x = \varphi \circ f \]

the pullback of \( \varphi \) along \( f \).
Definition (P-Russo)

Let \((A, \omega)\) and \((B, \xi)\) be \(C^*\)-algebras equipped with states. Let \(F : B \to A\) be a \(*\)-homomorphism such that the diagram on the right commutes.

\[
\begin{array}{ccc}
A & \xleftarrow{F} & B \\
\downarrow{\omega} & & \downarrow{\xi} \\
C & \xrightarrow{id} & B
\end{array}
\]

The latter diagram signifies commutativity \(\xi\)-a.e.
Non-commutative disintegrations

Definition (P-Russo)

Let \((\mathcal{A}, \omega)\) and \((\mathcal{B}, \xi)\) be C\(^*\)-algebras equipped with states. Let \(F : \mathcal{B} \to \mathcal{A}\) be a \(*\)-homomorphism such that the diagram on the right commutes.

A \textit{disintegration} of \(\omega\) over \(\xi\) consistent with \(F\) is a completely positive map \(R : \mathcal{A} \sim \to \mathcal{B}\) such that

\[
\begin{align*}
\mathcal{A} & \sim \to \mathcal{B} \\
\mathcal{B} & \sim \to \mathcal{B}
\end{align*}
\]

and

the latter diagram signifying commutativity \(\xi\)-a.e.
Existence and uniqueness of disintegrations

Surprising: existence is not guaranteed in the non-commutative setting!
Existence and uniqueness of disintegrations

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**Theorem (P-Russo)**

*Fix $n, p \in \mathbb{N}$. Let

$$\begin{align*}
\mathcal{M}_{np}(\mathbb{C}) & \xleftarrow{F} \mathcal{M}_n(\mathbb{C}) \\
\text{tr}(\rho \cdot) \equiv \omega & \quad \xi \equiv \text{tr}(\sigma \cdot) \\
\mathbb{C} &
\end{align*}$$

be a commutative diagram with $F$ the $\ast$-homomorphism given by the block diagonal inclusion $F(A) = \text{diag}(A, \ldots, A)$.\*
Existence and uniqueness of disintegrations

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Fix $n, p \in \mathbb{N}$. Let

$$M_{np}(\mathbb{C}) \xleftarrow{F} M_n(\mathbb{C})$$

$$\text{tr}(\rho \cdot) \equiv \omega$$

$$\xi \equiv \text{tr}(\sigma \cdot)$$

be a commutative diagram with $F$ the $*$-homomorphism given by the block diagonal inclusion $F(A) = \text{diag}(A, \ldots, A)$. A disintegration of $\omega$ over $\xi$ consistent with $F$ exists if and only if there exists a density matrix $\tau \in M_p(\mathbb{C})$ such that $\rho = \tau \otimes \sigma$. 
Example 1: Einstein-Podolsky-Rosen

Theorem (P-Russo)

Let

\[
\rho := \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \& \quad \sigma := \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
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and let \( F : \mathcal{M}_2(\mathbb{C}) \rightarrow \mathcal{M}_4(\mathbb{C}) \) be the diagonal map.
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0 & 1 \\
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

and let \( F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C}) \) be the diagonal map. Then \( \text{tr}(\sigma A) = \text{tr}(\rho F(A)) \) for all \( A \) but there does not exist a disintegration of \( \rho \) over \( \sigma \) consistent with \( F \).
Example 1: Einstein-Podolsky-Rosen

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and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ be the diagonal map. Then

$$\text{tr}(\sigma A) = \text{tr}(\rho F(A))$$

for all $A$ but there does not exist a disintegration of $\rho$ over $\sigma$ consistent with $F$.

Proof.

$\rho$ is entangled (not separable) and therefore cannot be expressed as the tensor product of any two $2 \times 2$ density matrices.
Example 2: Diagonal density matrices

Theorem (P-Russo)

Fix \( p_1, p_2, p_3, p_4 \geq 0 \) with \( p_1 + p_2 + p_3 + p_4 = 1 \), \( p_1 + p_3 > 0 \), and \( p_2 + p_4 > 0 \). Let

\[
\rho = \begin{bmatrix}
  p_1 & 0 & 0 & 0 \\
  0 & p_2 & 0 & 0 \\
  0 & 0 & p_3 & 0 \\
  0 & 0 & 0 & p_4 \\
\end{bmatrix}
\]

\&

\[
\sigma = \begin{bmatrix}
  p_1 + p_3 & 0 \\
  0 & p_2 + p_4 \\
\end{bmatrix}
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be density matrices and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ be the block diagonal inclusion. Then $\text{tr}(\sigma A) = \text{tr}(\rho F(A))$ for all $A$. Furthermore, there exists a disintegration of $\rho$ over $\sigma$ consistent with $F$ if and only if

$$
p_1 p_4 = p_2 p_3.
$$
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Fix $p_1, p_2, p_3, p_4 \geq 0$ with $p_1 + p_2 + p_3 + p_4 = 1$, $p_1 + p_3 > 0$, and $p_2 + p_4 > 0$. Let

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    p_1 & 0 & 0 & 0 \\
    0 & p_2 & 0 & 0 \\
    0 & 0 & p_3 & 0 \\
    0 & 0 & 0 & p_4 
\end{bmatrix}
&
\sigma = \begin{bmatrix}
    p_1 + p_3 & 0 \\
    0 & p_2 + p_4 
\end{bmatrix}
$$

be density matrices and let $F : \mathcal{M}_2(\mathbb{C}) \to \mathcal{M}_4(\mathbb{C})$ be the block diagonal inclusion. Then $\text{tr}(\sigma A) = \text{tr}(\rho F(A))$ for all $A$. Furthermore, there exists a disintegration of $\rho$ over $\sigma$ consistent with $F$ if and only if

$$p_1 p_4 = p_2 p_3.$$
Summary

Formulating concepts in probability theory categorically enables one to abstract these concepts to contexts beyond their initial domain.
Summary

Formulating concepts in probability theory categorically enables one to abstract these concepts to contexts beyond their initial domain. However, we still lack a full categorical probability theory. Amazing discoveries are yet to be made!
Thank you!

Thank you for your attention!